Exam multivariable analysis Jan 2021

Exercise 1

a. For $L \in \text{Hom}(V, W)$ and $w \in W$ define $f : V \to W$ by $f(v) = L(v) + w$. Use the definition of derivative to determine $f'(p)$ for any $p \in P$.

b. Take $V = \mathbb{R}^2$ and $W = \mathbb{R}$ and f as in part a). Use the properties of Theorem 2.1.1 to find an expression for $H'(p)(v)$ in terms of L, f, p, v and sin, cos, where $H : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $H(p) = \sin\Big(f\big(\sin(f(p)), \sin(f(p))\big)\Big)$. You may use $\sin' = \cos$ without proof.

Exercise 2 This question is about the inverse function theorem (Theorem 3.2.1 in the notes) and its proof.

a. Give an example of $f : P \to V$ and a $z \in P$ satisfying the conditions of the theorem such that the set A cannot be taken to be equal to P.

b. Explain what is meant in the second to last line of the proof *"In the final limit we used..."* by providing the missing details and explicitly prove the final equality $\lim_{h\to 0} \frac{Err(h)}{|h|} = 0$.

Exercise 3 The set of solutions to the system of two equations given below is called $S \subset \mathbb{R}^4$.

$$-xyz + 3z^3 - w^4 - 1 = 0$$
$$2xz - yw + x^3 - 2 = 0$$

a. Find a basis for the tangent space $T_{s_0}S$ to the solution $s_0 = (1, 1, 1, 1)$.

b. Use the implicit function theorem to show that close to the solution $x = y = z = w = 1$ the points of S can be written as C^1 functions of two out of the four variables.

Exercise 4

Suppose V is a vector space of dimension $2n$ and $\omega \in \Omega_1^n(V)$, where $n > 5$. For a fixed non-zero vector $w \in V$ consider the vector field $F : V \to V$ defined by $F(v) = w$. Also fix $p \in P$.

a. Check that $\alpha : (-1, 1) \to V$ given by $\alpha(t) = tw + p$ is an integral curve of F through p and compute $\alpha^*\omega$.

b. Take $\phi \in (\mathbb{R}^n)^*$ such that $|\phi(x)| < 1$ for all $x \in [0, 1]^n$. Prove that if the n-cube $\gamma : [0, 1]^n \to V$ satisfies $\gamma(x) = \alpha(\phi(x))$ for all $x \in [0, 1]^n$ then $\int_\gamma \omega = 0$.

Exercise 5 For $\omega \in \Omega_1^2(\mathbb{R}^4)$ and γ a 2-cube given by $\gamma(s, t) = (\sin 2\pi s, \cos 2\pi t, \cos 2\pi s, \sin 2\pi t)$.

a. Prove that $\int_\gamma \omega = 0$. using dω=0

b. Consider $\eta \in \Omega_1^2(\mathbb{R}^4)$ given by $\eta(x, y, z, w) = w\epsilon^1 \wedge \epsilon^2$ and express the pull-back $\gamma^*\eta$ in terms $ds \wedge dt$.

Exercise 6. Imagine a finite-dimensional vector space V and $s \in V$ and $\phi \in V^*$ such that $\phi(s) = 0$ and consider the shear map $S_{s,\phi} \in \text{Hom}(V, V)$.

a. Prove that if we define $\tilde{s} \in (V^*)^*$ by $\tilde{s}(f) = f(s)$ for any $f \in V^*$, then we have

$$(S_{s,\phi})^* = S_{\phi,\tilde{s}} \in \text{Hom}(V^*, V^*)$$

b. Is it true that for all $M \in \Lambda^3 V$ we have $\Lambda^3 S_{s,\phi} M = M$? Prove or provide a counter-example.